JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 139, 465–476 (1989)

Some Remarks on Nonlinear Second-Order Differential Equations with Periodic Boundary Conditions

JUKKA SARANEN AND SEPPO SEIKKALA

Section of Mathematics, Faculty of Technology, University of Oulu, Linnanmaa, SF-90570 Oulu, Finland

Submitted by V. Lakshmikantham

Received October 5, 1987

Here we propose a simple integral equation method for solution of nonlinear periodic boundary value problems. Our integral equation solves an auxiliary periodic boundary value problem with a control variable. The method is constructive; some numerical experiments are presented. © 1989 Academic Press, Inc.

1. INTRODUCTION

The nonlinear periodic boundary value problem

$$\begin{gathered} u''(t)=f(t, u(t), u'(t)), \qquad a<t<b, \\ u(a)=u(b), \qquad u'(a)=u'(b) \end{gathered} \tag{1.1}$$

has been studied extensively. The main techniques have been the method of upper and lower solutions, the use of Leray–Schauder topological degree, the variational approach, and monotone iterations [3, 5, 2, 8, 4]. Usually the function f is assumed to satisfy some additional conditions such as periodicity, monotonicity, Nagumo conditions, etc. Instead of these assumptions we use a Lipschitz condition on f and consider the more general problem

$$\begin{gathered} u''(t)=f(t, \mathscr{A}u(t), \mathscr{B}u'(t)), \qquad a<t<b, \\ u(a)=u(b), \qquad u'(a)=u'(b), \end{gathered} \tag{1.2}$$

where $\mathscr{A}, \mathscr{B}: L_2(a, b) \to L_2(a, b)$ are Lipschitz continuous. Furthermore we assume that the mapping $t \to f(t, v(t), w(t))$ is square integrable for any $v, w \in L_2(a, b)$. Here we apply our method used for Neumann-type problems in [7] and study the parametrized problem

0022-247X/89 $3.00

$$\begin{gathered} u''(t)+\delta=f(t, \mathscr{A}u(t), \mathscr{B}u'(t)) \\ u(a)=u(b), \qquad u'(a)=u'(b) \\ Ju=\lambda, \end{gathered} \tag{1.3}$$

where λ is a real number, J is the mean value operator

$$Ju=\frac{1}{b-a}\int_a^b u(t)\,dt,$$

and

$$\delta=Jf(\cdot, \mathscr{A}u(\cdot), \mathscr{B}u'(\cdot)). \tag{1.4}$$

We transform the problem (1.3) into an equivalent integral equation

$$u(t)=\lambda+\int_a^b k(t,s)\,f(s, \mathscr{A}u(s), \mathscr{B}u'(s))\,ds \tag{1.5}$$

and show the unique solvability of (1.5) by means of contraction mapping principle. Since every solution of (1.2) is obtained from (1.3) if and only if δ vanishes, the problem is to find the zeros of the mapping $\lambda\to\delta(\lambda)$. Under some additional conditions on f, similar to [6] we are able to show existence of a zero of $\delta(\lambda)$. Also we give examples, where the problem is solved numerically with Picard iterations.

The use of operators $\mathscr{A}$ and $\mathscr{B}$ in (1.2) allows us to discuss, e.g., third-order problems.

2. Main Results

We use the Sobolev spaces

$$H^1(a,b)=\{u\in L_2(a,b)\mid u'\in L_2(a,b)\}$$

and

$$H^2(a,b)=\{u\in H^1(a,b)\mid u''\in L_2(a,b)\}.$$

By a solution of the parametrized problem (1.3) we mean any function $u\in H^2(a,b)$ satisfying (1.3) for given $\lambda\in\mathbf{R}$.

Using the integral representation

$$\begin{aligned} u(t)=\tfrac{1}{2}\int_a^b |t-s|\,u''(s)\,ds+\tfrac{1}{2}[u(a)+u(b)] \\ -\tfrac{1}{2}[(a-t)u'(a)+(b-t)u'(b)], \qquad a\leqslant t\leqslant b, \end{aligned} \tag{2.1}$$

for $u \in H^2(a, b)$ we can derive an equivalent integral equation formulation of problem (1.3).

LEMMA 2.1. *A function $u \in H^1(a, b)$ is a solution of problem (1.3) if and only if u is a solution of the integral equation*

$$u(t) = \lambda + \int_a^b k(t, s) f(s, \mathscr{A}u(s), \mathscr{B}u'(s))\, ds, \qquad a \leqslant t \leqslant b, \tag{2.2}$$

where

$$k(t, s) = \frac{1}{2}\left\{|t-s| - \frac{1}{b-a}(t-s)^2 - \frac{b-a}{6}\right\}, \qquad (t, s) \in [a, b] \times [a, b]. \tag{2.3}$$

Proof. The kernel $k(t, s)$ is so chosen that the integral operator

$$\mathscr{K}u(t) = \int_a^b k(t, s) f(s, \mathscr{A}u(s), \mathscr{B}u'(s))\, ds \tag{2.4}$$

satisfies the periodic boundary conditions

$$\mathscr{K}u(a) = \mathscr{K}u(b), \qquad (\mathscr{K}u)'(a) = (\mathscr{K}u)'(b)$$

and has zero mean value for any $u \in H^1(a, b)$. Otherwise the argumentation follows the proof of Lemma 2.2 in [7] and is therefore omitted here. ∎

In order to prove the unique solvability of the problem (1.3) we assume that f, $\mathscr{A}$, and $\mathscr{B}$ satisfy the Lipschitz conditions

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \leqslant K\,|x - \bar{x}| + L\,|y - \bar{y}|, \tag{2.5}$$

$a \leqslant t \leqslant b$, $x, \bar{x}, y, \bar{y} \in \mathbf{R}$,

$$\|\mathscr{A}u - \mathscr{A}\bar{u}\| \leqslant A_0\,\|u - \bar{u}\|, \qquad u, \bar{u} \in L_2(a, b), \tag{2.6}$$

and

$$\|\mathscr{B}u - \mathscr{B}\bar{u}\| \leqslant B_0\,\|u - \bar{u}\|, \qquad u, \bar{u} \in L_2(a, b). \tag{2.7}$$

THEOREM 2.1. *If*

$$\frac{1}{4\pi^2}(b-a)^2 K^2 A_0^2 + L^2 B_0^2 < 2\pi^2/(b-a)^2, \tag{2.8}$$

then problem (1.3) has a unique solution for each $\lambda \in \mathbf{R}$.

Proof. The integral operator $\mathscr{K}$ given by (2.4) defines an operator from $H^1(a, b)$ into itself. Thus it suffices to show that $\mathscr{K}$ is a contraction in $H^1(a, b)$. Define the linear operators $\mathscr{K}_0, \mathscr{K}_1 : L_2(a, b) \to L_2(a, b)$ by

$$\mathscr{K}_0 u(t) = \int_a^b k(t, s) u(s)\, ds$$

and

$$\mathscr{K}_1 u(t) = \int_a^b k_t(t, s) u(s)\, ds.$$

These operators are bounded with operator norms

$$\|\mathscr{K}_0\| = \frac{1}{4\pi^2}(b-a)^2 \tag{2.9}$$

and

$$\|\mathscr{K}_1\| = \frac{1}{2\pi}(b-a). \tag{2.10}$$

Equations (2.9), (2.10) follow by calculating the smallest, in absolute value, eigenvalues of the compact selfadjoint operators $\mathscr{K}_0$ and $i\mathscr{K}_1$ in $L_2(a, b)$.

Now we have

$$\begin{aligned} \|\mathscr{K}u - \mathscr{K}\bar{u}\|^2 &\leqslant \|\mathscr{K}_0\|^2 \, \|f(\cdot, \mathscr{A}u(\cdot), \mathscr{B}u'(\cdot)) - f(\cdot, \mathscr{A}\bar{u}(\cdot), \mathscr{B}\bar{u}'(\cdot))\|^2 \\ &\leqslant 2\, \|\mathscr{K}_0\|^2 \, (K^2 \, \|\mathscr{A}u - A\bar{u}\|^2 + L^2 \, \|\mathscr{B}u' - \mathscr{B}\bar{u}'\|^2) \\ &\leqslant 2\, \|\mathscr{K}_0\|^2 \, (K^2 A_0^2 \, \|u - \bar{u}\|^2 + L^2 B_0^2 \, \|u' - \bar{u}'\|^2). \end{aligned} \tag{2.11}$$

Similarly, since $(\mathscr{K}_0 u)'(t) = (\mathscr{K}_1 u)(t)$, we have

$$\|(\mathscr{K}u)' - (\mathscr{K}\bar{u})'\|^2 \leqslant 2\, \|\mathscr{K}_1\|^2 \, (K^2 A_0^2 \, \|u - \bar{u}\|^2 + L^2 B_0^2 \, \|u' - \bar{u}'\|^2). \tag{2.12}$$

By choosing in $H^1(a, b)$ the norm

$$\|u\|_e = (K^2 A_0^2 \, \|u\|^2 + L^2 B_0^2 \, \|u'\|^2)^{1/2}$$

we obtain from (2.11)–(2.12)

$$\|(\mathscr{K}u) - (\mathscr{K}\bar{u})\|_e \leqslant \sqrt{2}\, (\|\mathscr{K}_0\|^2 \, \|K^2 A_0^2 + \|\mathscr{K}_1\|^2 \, L^2 B_0^2)^{1/2} \, \|u - \bar{u}\|_e. \tag{2.13}$$

From (2.8), (2.9), (2.10), and (2.13) it follows that $\mathscr{K}$ is a contraction in $H^1(a, b)$. ∎

If the condition (2.8) is valid, then Eq. (1.4),

$$\delta(\lambda) = Jf(\cdot, \mathscr{A}u_\lambda(\cdot), \mathscr{B}u'_\lambda(\cdot)),$$

u_λ being the unique solution of (1.3), defines a real function δ. The original problem (1.2) is solvable if and only if δ has a zero. For the function $\lambda \to (\delta(\lambda), u_\lambda)$ we have

COROLLARY 2.1. *Let* (2.8) *be valid. Then the mapping* $\lambda \to (\delta(\lambda), u_\lambda)$: $\mathbf{R} \to \mathbf{R} \times H^2(a, b)$ *is Lipschitz continuous. Accordingly, problem* (1.2) *has a solution if* $\delta(\lambda_1)\delta(\lambda_2) \leqslant 0$ *for some* $\lambda_1, \lambda_2 \in \mathbf{R}$.

Proof. For λ, $\bar{\lambda} \in \mathbf{R}$ we have

$$\begin{aligned}
|\delta(\lambda) - \delta(\bar{\lambda})| &\leqslant \frac{1}{b-a} \int_a^b |f(s, \mathscr{A}u_\lambda(s), \mathscr{B}u'_\lambda(s)) - f(s, \mathscr{A}u_{\bar{\lambda}}(s), \mathscr{B}u'_{\bar{\lambda}}(s))| \, ds \\
&\leqslant \frac{1}{b-a} \int_a^b (K\, |\mathscr{A}u_\lambda(s) - \mathscr{A}u_{\bar{\lambda}}(s))| + L\, |\mathscr{B}u'_\lambda(s) - \mathscr{B}u'_{\bar{\lambda}}(s)|)\, ds \\
&\leqslant \frac{1}{b-a} \sqrt{b-a}\, (KA_0\, \|u_\lambda - u_{\bar{\lambda}}\| + LB_0\, \|u'_\lambda - u'_{\bar{\lambda}}\|) \\
&\leqslant \frac{\sqrt{2}}{\sqrt{b-a}}\, \|u_\lambda - u_{\bar{\lambda}}\|_e
\end{aligned}$$

and by (2.13)

$$\begin{aligned}
\|u_\lambda - u_{\bar{\lambda}}\|_e &\leqslant \|\lambda - \bar{\lambda}\|_e + \|\mathscr{K}u_\lambda - \mathscr{K}u_{\bar{\lambda}}\|_e \\
&\leqslant KA_0 \sqrt{b-a}\, |\lambda - \bar{\lambda}| + c\, \|u_\lambda - u_{\bar{\lambda}}\|_e,
\end{aligned}$$

where

$$c = \sqrt{2}\, (\|\mathscr{K}_0\|^2\, K^2 A_0^2 + \|\mathscr{K}_1\|^2\, L^2 B_0^2)^{1/2} < 1.$$

Hence

$$\|u_\lambda - u_{\bar{\lambda}}\|_e \leqslant \frac{KA_0 \sqrt{b-a}}{1-c}\, |\lambda - \bar{\lambda}| \tag{2.14}$$

and

$$|\delta(\lambda) - \delta(\bar{\lambda})| \leqslant \frac{\sqrt{2}\, KA_0}{1-c}\, |\lambda - \bar{\lambda}|. \tag{2.15}$$

Finally, we have

$$\begin{aligned}\|u''_\lambda - u''_{\bar\lambda}\| &\leqslant \|\delta(\lambda) - \delta(\bar\lambda)\| \\ &\quad + \|f(\cdot, \mathscr{A}u_\lambda(\cdot), \mathscr{B}u'_\lambda(\cdot)) - f(\cdot, \mathscr{A}u_{\bar\lambda}(\cdot), \mathscr{B}u'_{\bar\lambda}(\cdot))\| \\ &\leqslant \sqrt{b-a}\,|\lambda - \bar\lambda| + \sqrt{2}\,\|u_\lambda - u_{\bar\lambda}\|_e. \end{aligned} \tag{2.16}$$

The assertions follow by (2.14)–(2.16). ∎

As an example we consider the case where $\mathscr{A} = \mathscr{B} = I$ and the function f satisfies the following conditions. For related assumptions see, e.g., [4, 6].

(i) f is uniformly essentially bounded in t such that

$$|f(t, x, y)| \leqslant M, \qquad x, y \in \mathbf{R}, \quad \text{a.e. } t \in [a, b],$$

(ii) the limits

$$f(t, \pm\infty, y_0) = \lim_{\substack{x \to \pm\infty \\ y \to y_0}} f(t, x, y)$$

exist for a.e. $t \in [a, b]$ and

(iii) $\int_a^b f(t, \infty, w(t))\, dt < 0$, $\int_a^b f(t, -\infty, w(t))\, dt > 0$ for all continuous functions w such that $|w(t)| \leqslant N$, $a \leqslant t \leqslant b$, where

$$N = M \max_{a \leqslant t \leqslant b} \int_a^b |k_t(t, s)|\, ds.$$

COROLLARY 2.2. *Assume that f satisfies the assumption of Theorem* 2.1 *and conditions* (i)–(iii) *above. Then problem* (1.2) *has a solution.*

Proof. By Corollary 2.1 it suffices to show that $\delta(\lambda_1)\delta(\lambda_2) \leqslant 0$ for some λ_1, λ_2. From

$$u_\lambda(t) = \lambda + \int_a^b k(t, s)\, f(s, u_\lambda(s), u'_\lambda(s))\, ds$$

and

$$u'_\lambda(t) = \int_a^b k_t(t, s)\, f(s, u_\lambda(s), u'_\lambda(s))\, ds$$

it follows that

$$u_\lambda(t) \to \pm\infty, \qquad \lambda \to \pm\infty$$

and

$$|u'_\lambda(t)| \leqslant N, \qquad (t, \lambda) \in [a, b] \times \mathbf{R}.$$

Furthermore

$$u_\lambda''(t) = f(t, u_\lambda(t), u_\lambda'(t)) - Jf(\cdot, u_\lambda(\cdot), u_\lambda'(\cdot))$$

yields

$$\|u_\lambda''\| \leqslant 2\sqrt{b-a}\, M.$$

Since the embedding $H^1(a, b) \subset L_2(a, b)$ is compact, we can choose an in $L_2(a, b)$ convergent sequence u'_{λ_k}, $\lambda_k \to \infty$, such that $u'_{\lambda_k}(t) \to w(t)$ a.e. By (ii) and by the Lebesgue convergence theorem we obtain

$$\begin{aligned}\delta(\lambda_k) &= \frac{1}{b-a}\int_a^b f(t, u_{\lambda_k}(t), u'_{\lambda_k}(t))\, dt \\ &\to \frac{1}{b-a}\int_a^b f(t, \infty, w(t))\, dt < 0.\end{aligned} \tag{2.17}$$

Similarly we find $\lambda_k \to -\infty$ with

$$\delta(\lambda_k) \to \frac{1}{b-a}\int_a^b f(t, -\infty, \bar{w}(t))\, dt > 0. \tag{2.18}$$

Results (2.17), (2.18) imply the assertion. ∎

In case $f(t, x, y)$ is periodic in x we get

COROLLARY 2.3. *Let $\mathscr{A}u = \mathscr{B}u = u$ and let the assumptions of Theorem* 2.1 *be valid. If $f(t, x, y)$ is p-periodic in x for all $t \in [a, b]$, $y \in \mathbf{R}$, then $\delta(\lambda)$ is p-periodic and hence problem* (1.2) *has either infinitely many solutions or no solutions.*

For the proof cf. [7, Corollary 2.3].

3. Some Examples and Remarks

The appearance of the operators $\mathscr{A}$ and $\mathscr{B}$ allows us to discuss some third-order differential equations.

EXAMPLE 3.1. We consider the third-order problem

$$\begin{aligned} & w'''(t) = f(t, w, w''), \qquad a < t < b \\ & w(a) = w_a \\ & w'(a) = w'(b) \\ & w''(a) = w''(b). \end{aligned} \tag{3.1}$$

Introducing $u(t)=w'(t)$, we obtain

$$\begin{aligned} &u''(t)=f(t, \mathscr{A}u(\cdot), \mathscr{B}u'(\cdot)), \qquad a<t<b \\ &u(a)=u(b) \\ &u'(a)=u'(b), \end{aligned} \tag{3.2}$$

where

$$\mathscr{A}u(t)=w_a+\int_a^t u(s)\,ds,$$
$$\mathscr{B}u'(t)=u'(t).$$

For the operator $\mathscr{A}$ we may choose

$$A_0=\frac{1}{\sqrt{2}}(b-a).$$

EXAMPLE 3.2. This is an example of the case where the assumptions of Corollary 2.2 are valid, but where f is not periodic and not of the form $f=p(t)-g(x, y)$ as in [4]. Let

$$f(t, x, y)=p(t)-e^{-|x|}\sin y-f(x)[h(t)+|\sin y|], \qquad 0\leqslant t\leqslant 1 \quad x, y\in \mathbf{R},$$

where

$$f(x)=\begin{cases} x, & |x|\leqslant 1 \\ \operatorname{sgn} x, & |x|>1, \end{cases}$$

p and h are square integrable and essentially bounded on $[0, 1]$, h is non-negative,

$$\left|\int_0^1 p(t)\,dt\right|<\int_0^1 h(t)\,dt,$$

and

$$\operatorname{ess\,sup} h(t)<2\pi\sqrt{2\pi^2-4}-2.$$

Now conditions (i)–(iii) and inequality (2.8) are satisfied.

Remark 3.1. Condition (2.8) is not optimal in all cases. For example, if f on the right-hand side of (1.2) is independent of u', then we have $L=0$ and (2.8) can be replaced by the optimal condition $KA_0<4\pi^2/(b-a)^2$.

Remark 3.2. Instead of (2.3) one can consider the parametrized problem

$$
\begin{aligned}
u''(t) &= f(t, \mathscr{A}u(\cdot), \mathscr{B}u'(\cdot)) \\
u(a) &= u(b) = \lambda \in \mathbf{R}
\end{aligned}
\tag{3.3}
$$

with the control

$$\delta(\lambda) = Jf(\cdot, \mathscr{A}u(\cdot), \mathscr{B}u'(\cdot)) = u'_\lambda(b) - u'_\lambda(a) \tag{3.4}$$

(cf. [1]). This approach leads to the integral equation

$$u(t) = \lambda + \int_a^b g(t, s)\, f(s, \mathscr{A}u(s), \mathscr{B}u'(s))\, ds, \tag{3.5}$$

where $g(t, s)$ is the Green function of the Dirichlet problem. Now the linear integral operator $G_0 \colon L_2(a, b) \to L_2(a, b)$ given by this kernel has the norm

$$\|G_0\| = \frac{1}{\pi^2}(b-a)^2. \tag{3.6}$$

Thus the use of (1.3) with the integral operator $\mathscr{K}_0$ leads to a better constraint (2.8).

Remark 3.3. The above method can be used also in some cases where the function f is not globally Lipschitzean. For example, if it is known that the solution is in some ball, it suffices that f is locally Lipschitzean. This is illustrated in the following example.

EXAMPLE 3.3. We consider periodic solutions to the forced Duffing equation

$$u''(t) + \alpha u(t) + \beta u^3(t) = \kappa \cos 2\pi t, \qquad 0 < t < 1, \tag{3.7}$$

in which α, β, κ are real constants and $\alpha > 0$. If the solution u is bounded by $|u(t)| \leqslant M$, $0 \leqslant t \leqslant 1$, then u satisfies the equation

$$u''(t) = f(t, u(t)), \qquad 0 < t < 1, \tag{3.8}$$

where

$$f(t, x) = \begin{cases} \kappa \cos 2\pi t - \alpha x - \beta x^3, & |x| \leqslant M, \\ \kappa \cos 2\pi t + \alpha M + \beta M^3, & x < -M, \\ \kappa \cos 2\pi t - \alpha M - \beta M^3, & x > M. \end{cases}$$

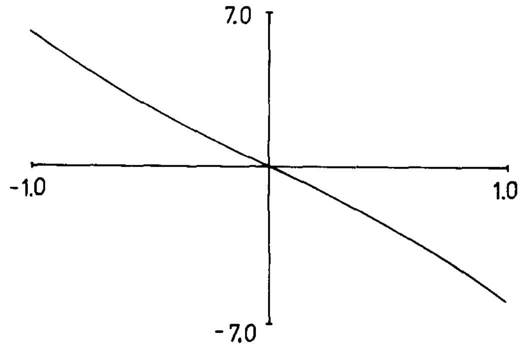

FIG. 1. Behavior of $\delta(\lambda)$ on the interval $[-1, 1]$.

The function f is Lipschitz continuous with the Lipschitz constant $K = \alpha + 3M^2 |\beta|$. By Remark 3.1 we can apply our results to Eq. (3.8) if

$$\alpha + 3M^2 |\beta| < 4\pi^2. \tag{3.9}$$

For numerical calculations we have chosen $\alpha = 5$, $\beta = 1$, $\kappa = 7$, $M = 2$. Figure 1 shows the behavior of $\delta(\lambda)$ on the interval $[-1, 1]$.

We find the zero of $\delta(\lambda)$ at $\lambda_0 \cong 0$. The corresponding approximate solution u_0 is shown in Fig. 2.

EXAMPLE 3.4. Now we consider a slight modification of the Duffing-type equation, viz.,

$$u''(t) + e^{-|u'(t)|}u(t) + 5u(t) \sin 8u(t) = 10t \cos 2\pi t, \qquad 0 < t < 1. \tag{3.10}$$

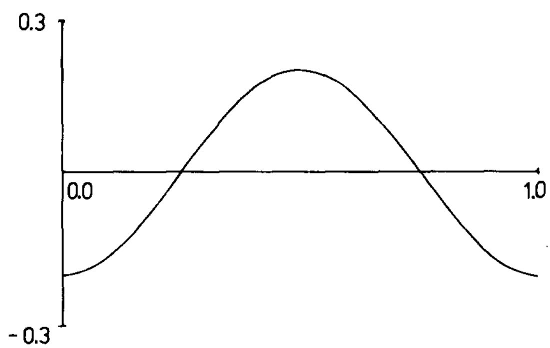

FIG. 2. Function u_0.

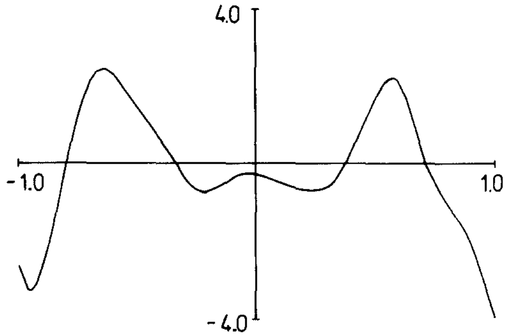

FIG. 3. Behavior of $\delta(\lambda)$ on the interval $[-1, 1]$.

Observe that in this example $f(t, x, y)$ is not periodic in t. We found numerically the zeros $\lambda_1 \simeq -0.79484$, $\lambda_2 \simeq -0.33870$, $\lambda_3 \simeq 0.38225$, and $\lambda_4 \simeq 0.71339$ of $\delta(\lambda)$ (see Fig. 3). The graphs of the corresponding periodic solutions u_1, u_2, u_3, and u_4 of (3.10) are given in Fig. 4.

EXAMPLE 3.5. We point out that our results apply also in the case where the function $f(t, u, u')$ is not continuous. As an example consider the equation

$$u''(t) = 6u(t) + \sin[u'(t)] + h(t), \qquad -1 < t < 1, \tag{3.11}$$

where

$$h(t) = \begin{cases} 8t^3 - 6t^2 - 21t + 2 - \sin(-4t^2 + 2t + \frac{13}{6}), & -1 \leqslant t < 0, \\ 12t^3 - 25t - \sin(-6t^2 + \frac{13}{6}), & 0 < t \leqslant 1. \end{cases}$$

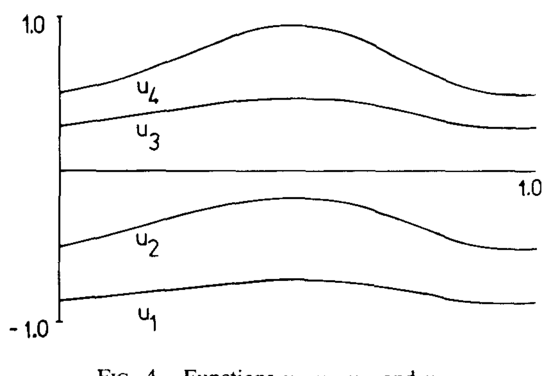

FIG. 4. Functions u_1, u_2, u_3, and u_4.

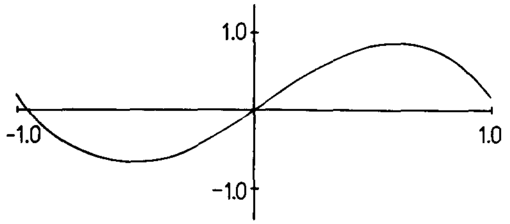

FIG. 5. Behavior of the solution u_0.

We find a zero $\lambda_0 \cong 0.0862$. The corresponding solution u_0 is shown in Fig. 5. In all calculations above we have used the composite trapezoidal rule with sixty subintervals for numerical integration. For a zero λ_0 of $\delta(\lambda)$ we have the accuracy $|\delta(\lambda_0)| \leqslant 10^{-4}$.

REFERENCES

1. S. R. BERNFELD AND V. LAKSHMIKANTHAM, "An Introduction to Boundary Value Problems," Academic Press, New York, 1974.
2. P. G. CIARLET, M. H. SCHULTZ, AND R. S. VARGA, Numerical methods of high-order accuracy for nonlinear boundary value problems. IV. Periodic boundary conditions, *Numer. Math.* **12** (1968), 266–279.
3. G. FOURNIER AND J. MAWHIN, On periodic solutions of forced pendulum-like equations, *J. Differential Equations* **60** (1985), 381–395.
4. R. KANNAN AND R. ORTEGA, Periodic solutions of pendulum-type equations, *J. Differential Equations* **59** (1985), 123–144.
5. V. LAKSHMIKANTHAM AND S. LEELA, Remarks on first and second order periodic boundary value problems, *Nonlinear Anal.* **8** (1984), 281–287.
6. E. M. LANDESMAN AND A. C. LAZER, Nonlinear perturbations of linear elliptic boundary value problems at resonance, *J. Math. Mech.* **19** (1970), 609–623.
7. J. SARANEN AND S. SEIKKALA, Solution of a nonlinear two-point boundary value problem with Neumann-type boundary data, *J. Math. Anal. Appl.* **135** (1988), 691–701.
8. G. VIDOSSICH, One-sided existence and uniqueness for periodic solutions of second order ordinary differential equations, *Nonlinear Anal.* **10** (1986), 1123–1131.